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Birefringence by a smoothly inhomogeneous locally isotropic medium: Three-dimensional case

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The propagation problem for electromagnetic waves in a smoothly inhomogeneous locally isotropic medium, which was considered for a layered case in V. S. Liberman and B. Ya. Zel'dovich, Phys. Rev. E **49**, 2389 (1994) is generalized to a three-dimensional situation. Effective "linear" birefringence, i.e., coherent transformation of a right circularly polarized wave into the left one with the amplitude $\sim (\lambda/a)$ is predicted and calculated. It corresponds to the corrections $\delta n \sim (\lambda/a)^2$ to the effective refractive index tensor, where $a \gg \lambda$ is the size of smooth inhomogeneity. An important feature is that linear birefringence appears only in the presence of gradients of impedance $\rho(\mathbf{r}) = \sqrt{\mu(\mathbf{r})/\varepsilon(\mathbf{r})}$, whereas the gradients of refractive index $n(\mathbf{r}) = \sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})}$ are not necessary in a general three-dimensional case. This is in contrast with a layered medium (one-dimensional case) where the net effect was proportional to the product $(d \ln \rho/dz)(d \ln n/dz)$.

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I. INTRODUCTION

A novice's idea of geometric optics, "propagation of light along straight rays," deals with the case of a homogeneous medium only. Such an approach is useful if there are sharp boundaries between two regions (e.g., air and glass) so that the problem of solving Maxwell equations is reduced to matching two or three plane wave solutions with the use of Snell's law, Fresnel reflection, and transmission formulas. The situation is somewhat more complicated when the refractive index $n(\mathbf{r})$ of the medium smoothly varies in space. In that case the rays are deflected in a gradual way. Well-known ray equations are [1–3]

$$\frac{d\mathbf{r}}{dl} = \mathbf{s} , \tag{1}$$

$$\frac{d\mathbf{s}}{dl} = \nabla \ln n - \mathbf{s}\,(\mathbf{s}\cdot\nabla \ln n) , \tag{2}$$

where $\mathbf{s}$ is the direction vector of the photon momentum, $\mathbf{s} = \mathbf{p}/p$, $|\,\mathbf{s}\,| = 1$, and l is the length measured along the trajectory. Throughout the paper we shall consider the medium with locally isotropic dielectric ε_{ik} and magnetic μ_{ik} susceptibilities,

$$\varepsilon_{ik}(\mathbf{r}) = \varepsilon(\mathbf{r})\delta_{ik}, \qquad \mu_{ik}(\mathbf{r}) = \mu(\mathbf{r})\delta_{ik} , \tag{3}$$

and assume that the spatial size a of inhomogeneities in $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ is much larger than the wavelength $\lambda/2\pi$.

The condition $a \gg \lambda$ along with the corresponding assumptions about the properties of the incident waves constitute the requirements for the applicability of geometric optics, or the WKB approximation. In that approach the phase of a wave is accumulated along the trajectory with the rate $\omega n/c$, so that [1–3]

$$\varphi(\mathbf{r}) - \varphi(\mathbf{r}_0) = \frac{\omega}{c}\int_{l_0}^{l} n(\mathbf{r}(l'))dl' . \tag{4}$$

Taking $l - l_0$ about the inhomogeneity size, $l - l_0 \sim a$, one gets the estimation of the basic contribution (4) to the phase

$$[\varphi(\mathbf{r}) - \varphi(\mathbf{r}_0)]^{(b)} = (\lambda/a)^{-1} \gg 1 . \tag{5}$$

A natural question arises about a more accurate calculation of the wave's phase, i.e., about the corrections of the order $(\lambda/a)^0$ and $(\lambda/a)^1$ to the basic expression (4). Common opinion is that one should consider the three-dimensional Schrödinger-type (or Helmholtz) scalar wave equation

$$\Delta\Psi(\mathbf{r}) + \frac{\omega^2}{c^2}n^2(\mathbf{r})\Psi = 0 . \tag{6}$$

Then the well-known phase shift $\Delta\varphi = -\pi/2 \sim (\lambda/a)^0$ appears due to each passage of a ray near a caustic surface; the number of such passages is connected with the Maslov index of the ray's manifold [4, 5].

We would like to emphasize a fact that was understood at least by the beginning of the 20th century. Namely, the question about the phase with such precision is senseless for electromagnetic waves if one does not specify the polarization of the wave for which that phase is to be calculated (or measured). For the specific case of planar trajectory the transverse character of polarization allows us to choose two natural polarization unit vectors. One of them is $\mathbf{e}_\perp$, i.e., perpendicular to the plane of trajectory, and the other is $\mathbf{e}_\parallel = \mathbf{e}_\perp \times \mathbf{s}$, i.e., lying in the trajectory plane, but orthogonal to the propagation direction $\mathbf{s}$ at each point of the ray $\mathbf{r}(l)$. Here and in the following by polarization we mean the direction of the complex vector amplitude $\mathbf{E}(\mathbf{r})$ of the electric field

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$$\mathbf{E}_{\rm real}(\mathbf{r},t) = \frac{1}{2}\left[\,\mathbf{E}(\mathbf{r})\exp(-i\omega t) + \mathbf{E}^*(\mathbf{r})\exp(i\omega t)\,\right], \quad (7)$$

since in the visible range optics $\varepsilon - 1 \neq 1$ and $\mu - 1 = 0$. The choice of magnetic vector would be equally suitable for that purpose, but we will not use it due to our "optical roots."

It may be shown that the phase corrections $\sim (\lambda/a)^0$ for $E_{\parallel}$ and $E_{\perp}$ are just the same as for the scalar Helmholtz-Schrödinger equation (6), i.e., are connected with the passages past caustic surfaces and with Maslov index [4–6]. However, for the general case of a trajectory possessing helicity there is no evident choice of the two basic polarization vectors in a continuum of planes perpendicular to continuosly varying propagation direction $\mathbf{s}(l)$. There is an especially clear discussion of that fact in the paper [7]. The evolution law of the polarization vector was established in [8–10] (see also [1, 2]) and consists in a so-called "parallel transport:" let us try not to change the polarization vector $\mathbf{e}$ since the medium is locally isotropic; the only changes that we must introduce are those which keep the polarization transverse to the new propagation direction

$$\frac{d\mathbf{e}}{dl} = -\mathbf{s}\left(\mathbf{e}\cdot\frac{d\mathbf{s}}{dl}\right), \quad (8)$$

so that $(\mathbf{e}\cdot\mathbf{s}) \equiv 0$ at any l, if $(\mathbf{e}\cdot\mathbf{s})$ was equal to zero at the starting point.

In the Russian scientific literature that "parallel transport" evolution is called sometimes "Rytov's rotation" [9, 2], meaning some very particular (Frenet) choice of azimuthal position of coordinate system versus l. As clearly explained in [7], that choice is not always the best one (not to say anything derogatory about Rytov's work [9]).

Right circular polarization $\mathbf{e} = (\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ for $\mathbf{s} = \mathbf{e}_z$ stays of the same circular type during the propagation according to Eq. (8) and acquires some additional phase. Since the unit vector $(\mathbf{e}_x + i\mathbf{e}_y)/\sqrt{2}$ gets a phase factor $(\mathbf{e}'_x + i\mathbf{e}'_y) = \exp(-i\alpha)(\mathbf{e}_x + i\mathbf{e}_y)$ under the coordinate system rotation $\mathbf{e}'_x = \mathbf{e}_x\cos\alpha + \mathbf{e}_y\sin\alpha$, $\mathbf{e}'_y = -\mathbf{e}_x\sin\alpha + \mathbf{e}_y\cos\alpha$ at an angle α, the additional phase of circularly polarized wave depends explicitly on the particular choice of the frames continuum; see [7]. The additional phase factor for the left circular polarization also depends on that choice. Parallel transport Eq. (8) may be described as "circular birefringence" $\delta n_+ - \delta n_- \sim (\lambda/a)^1$, so that the change of polarization at a distance a is about 100 %, i.e., about 1 rad. However, the conservation of circularity type means that the input linear polarization stays linear in that approximation. In other words, initially real $\mathbf{e}$ stays real after the evolution according to Eq. (8). The invariant value of the rotation angle may be determined only in the case when $\mathbf{s}(l_0) = \mathbf{s}(l_1)$; that rotation angle is expressed via the geometrical Berry phase [11], i.e., via the phase difference for the two circularly polarized components; see also [8, 10, 12].

Berry's phase or parallel transport is the influence of the trajectory on the polarization. The back influence of circular polarization on the trajectory [13, 14] may be described by the modification of Eq. (1):

$$\frac{d\mathbf{r}}{dl} = \mathbf{s} + \frac{c}{\omega n}\sigma\left[\mathbf{s}\times\frac{d\mathbf{s}}{dl}\right] \quad (9)$$

and was called the "optical magnus effect" in [14]. Here $\sigma = +1$ for the right circular polarization and $\sigma = -1$ for the left one. Both effects may be considered as the consequence of a spin-orbit interaction of a transverse wave in an inhomogeneous medium [15].

Going back to the wave's phase, one comes to the conclusion that in the approximation taking into account $\delta n \sim (\lambda/a)^1$ terms, the WKB solution of the Maxwell equations is given by the corresponding solution of the scalar Helmholtz-Schrödinger equation (6) multiplied by the polarization vector $\mathbf{e}$ taken from Eq. (8). This statement will be proved explicitly below. An important feature of such an approximation is that it is determined by the profile of refractive index

$$n(\mathbf{r}) = \sqrt{\varepsilon(\mathbf{r})\mu(\mathbf{r})} \quad (10)$$

only and does not depend on ε and μ separately. In other words, that approximation deals with the geometry of the rays only and not with the electrodynamics in particular. For example, the transverse acoustical waves in locally isotropic smoothly inhomogeneous medium must show similar "geometric" properties. At first sight the calculation of higher order corrections $\delta\varphi \sim (\lambda/a)^1$ or $\delta n_{\rm eff} \sim (\lambda/a)^2$ seems to be a very dull job, since in almost any complicated case our knowledge of the particular profile of the refractive index is not very good and must introduce much larger error into the result. However, just for the transverse electromagnetic waves in a locally isotropic medium the main (large) part of the phase uncertainty due to poor knowledge of $n(\mathbf{r})$ is identical for two degenerate polarizations. Therefore a small effective "linear" birefringence $n_{\parallel} - n_{\perp} \sim (\lambda/a)^2$ gives a qualitatively new physical effect: partial transformation of right circular polarization into the left one, transformation of linear polarization into weakly elliptical polarization. Both those effects may be easily measured in optics.

Such a birefringence was calculated in [16] for a restricted case when both $\varepsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ depend on one coordinate z only—the so-called layered medium. It turned out that, contrary to the purely geometric "circular" birefringence $\delta n_+ - \delta n_- \sim (\lambda/a)$, the linear one is about $\delta n \sim (\lambda/a)^2$ and is determined by both refractive index $n(\mathbf{r})$ and impedance $\rho(\mathbf{r})$ profiles; here

$$\rho(\mathbf{r}) = \sqrt{\mu(\mathbf{r})/\varepsilon(\mathbf{r})}\ . \quad (11)$$

The present paper is devoted to the calculation of effective birefringence $\delta n \sim (\lambda/a)^2$ or $\delta\varphi \sim (\lambda/a)^1$ for a general three-dimensional problem. It turns out that extra dimensions introduce qualitatively new features to the resulting birefringence.

II. MAXWELL EQUATIONS FOR AN INHOMOGENEOUS MEDIUM IN TERMS OF HELICITY AMPLITUDES

We adopt the following system of Maxwell equations for complex amplitudes of monochromatic fields:

$$\mathrm{rot}\mathbf{E} = i\frac{\omega}{c}\mu\mathbf{H}(\mathbf{r}), \qquad \mathrm{rot}\mathbf{H} = -i\frac{\omega}{c}\varepsilon\mathbf{E}(\mathbf{r}) \tag{12}$$

so that the equalities $\mathrm{div}(\varepsilon\mathbf{E}) = \mathrm{div}(\mu\mathbf{H}) = 0$ are the consequences of (12). Useful combinations of those fields are

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\sqrt{\rho(\mathbf{r})}}\mathbf{E}(\mathbf{r}) + i\sqrt{\rho(\mathbf{r})}\mathbf{H}(\mathbf{r}) \ , \tag{13}$$

$$\mathbf{B}(\mathbf{r}) = \frac{1}{\sqrt{\rho(\mathbf{r})}}\mathbf{E}(\mathbf{r}) - i\sqrt{\rho(\mathbf{r})}\mathbf{H}(\mathbf{r}) \ . \tag{14}$$

It is worth noting that $\mathbf{E}$ and $\mathbf{H}$ are complex vectors and therefore $\mathbf{A} \neq \mathbf{B}^*$. Direct substitution of (13) and (14) into (12) yields

$$\mathrm{rot}\mathbf{A} = k(\mathbf{r})\mathbf{A} - \frac{1}{2}\mathbf{G}\times\mathbf{B} \ , \tag{15}$$

$$\mathrm{rot}\mathbf{B} = -k(\mathbf{r})\mathbf{B} - \frac{1}{2}\mathbf{G}\times\mathbf{A} \ . \tag{16}$$

Here and below we need two vectors

$$\mathbf{G} = \boldsymbol{\nabla}\ln\rho(\mathbf{r}), \qquad \mathbf{L} = \boldsymbol{\nabla}\ln n(\mathbf{r}) \ , \tag{17}$$

$$k(\mathbf{r}) = \frac{\omega}{c}n(\mathbf{r}) \ , \tag{18}$$

with c being the speed of light in vacuum. It is interesting to note that the Poynting vector $\mathbf{S} = (c/16\pi)(\mathbf{E}^*\times\mathbf{H} + \mathbf{E}\times\mathbf{H}^*)$ takes a very simple form in terms of $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{S} = i\frac{c}{32\pi}(\mathbf{B}^*\times\mathbf{B} - \mathbf{A}^*\times\mathbf{A}) \ . \tag{19}$$

If the medium is homogeneous, $\rho = \mathrm{const}$, $k = \mathrm{const}$, and $\mathbf{L} = \mathbf{G} \equiv \mathbf{0}$, then the solutions of the Eqs. (15) and (16) are very simple. For example, circularly polarized plane waves are

$$\exp(-i\omega t)\mathbf{A}(\mathbf{r}) = A_0\frac{(\mathbf{e}_x + i\mathbf{e}_y)}{\sqrt{2}}\exp(ikz - i\omega t) \ , \tag{20}$$

$$\exp(-i\omega t)\mathbf{B}(\mathbf{r}) = B_0\frac{(\mathbf{e}_x - i\mathbf{e}_y)}{\sqrt{2}}\exp(ikz - i\omega t) \ . \tag{21}$$

An arbitrary solution may be written as a superposition of such plane waves with various propagation directions [instead of $\mathbf{e}_z$ in (20) and (21)]. In particular, the solution $\mathbf{A}(\mathbf{r}) \neq \mathbf{0}$, $\mathbf{B}(\mathbf{r}) \equiv \mathbf{0}$ corresponds to purely right circularly polarized waves, similarly $\mathbf{A}(\mathbf{r}) \equiv \mathbf{0}$, $\mathbf{B}(\mathbf{r}) \neq \mathbf{0}$ is a left circular polarized wave.

If $k = k(\mathbf{r}) \neq \mathrm{const}$, then a plane wave suffers some scattering and deflection by inhomogeneities. We will consider the problem where $k \to \mathrm{const}$ and $\rho \to \mathrm{const}$ both in the regions of incident and deflected waves; those limiting values k_{inc} and k_{def} may be different, as well as ρ_{inc} and ρ_{def}. A remarkable feature of the Maxwell system (15) and (16) is that for a general inhomogeneous medium ($k \neq \mathrm{const}$) possessing constant impedance $\rho = \mathrm{const}$, an incident right polarized wave [$\mathbf{A}(\mathbf{r}) \neq \mathbf{0}$, $\mathbf{B}(\mathbf{r}) = \mathbf{0}$] will stay so with 100 % accuracy.

Birefringence may be considered as a transformation from one circular polarization into the other, $A \to B$ and vice versa $B \to A$. We are interested in a situation when such a birefringence is small, e.g., $\delta A \sim (\lambda/a)B$ or $\delta n_{\mathrm{eff}} \sim (\lambda/a)^2$. At first sight, the right-hand side of Eqs. (15) and (16) allows us to consider that mixing as an effect

$$\frac{\delta\hat{n}}{n} \sim \frac{G}{k} \sim \frac{\boldsymbol{\nabla}\ln\rho}{k} \sim (\lambda/a) \tag{22}$$

in contradiction with the expectations described in the Introduction. Here the caret over the $\hat{n}$ symbolizes the anisotropic contribution to $\delta\hat{\varepsilon}$ and $\delta\hat{\mu}$. Actually the estimation (22) is incorrect. To show that explicitly, we can take into account a consequence of Eqs. (15) and (16):

$$\mathrm{div}\mathbf{A} = \frac{1}{2}(\mathbf{GB}) - (\mathbf{LA}), \qquad \mathrm{div}\mathbf{B} = \frac{1}{2}(\mathbf{GA}) - (\mathbf{LB}) \ . \tag{23}$$

That allows us to get second-order equations for $\mathbf{A}$ and $\mathbf{B}$:

$$\Delta\mathbf{A} + k^2\mathbf{A} + 2\boldsymbol{\nabla}(\mathbf{LA}) - \frac{1}{4}\mathbf{G}^2\mathbf{A} - (\mathbf{L}\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\boldsymbol{\nabla})\mathbf{L}$$
$$= \frac{1}{2}\mathbf{B}(\mathbf{LG} - \mathrm{div}\mathbf{G}) + (\mathbf{B}\boldsymbol{\nabla})\mathbf{G} - (\mathbf{BL})\mathbf{G} \ , \tag{24}$$

$$\Delta\mathbf{B} + k^2\mathbf{B} + 2\boldsymbol{\nabla}(\mathbf{LB}) - \frac{1}{4}\mathbf{G}^2\mathbf{B} - (\mathbf{L}\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\boldsymbol{\nabla})\mathbf{L}$$
$$= \frac{1}{2}\mathbf{A}(\mathbf{LG} - \mathrm{div}\mathbf{G}) + (\mathbf{A}\boldsymbol{\nabla})\mathbf{G} - (\mathbf{AL})\mathbf{G} \ . \tag{25}$$

According to (24) and (25), the connection between two circular components $\mathbf{A}$ and $\mathbf{B}$ turns out to be the effect of the order

$$\frac{\delta\hat{n}}{n} \sim \frac{\boldsymbol{\nabla}\mathbf{G}}{k^2} \sim \frac{\mathbf{L}\cdot\mathbf{G}}{k^2} \sim (\lambda/a)^2 \ . \tag{26}$$

However, the particular calculation of the effective birefringence, or $B \to A$ mixing, will be done below with the use of the first-order system (15) and (16).

III. WKB OR GEOMETRIC OPTICS NOTATIONS FOR THE SOLUTION OF MAXWELL EQUATIONS

Here we shall introduce the notation of the WKB approximation or geometric optics which will be used for the asymptotic solution of Maxwell equations with the necessary accuracy; see, e.g., [17,1,3,6]. With this aim in the mind, we will consider the real eikonal function $\psi(\mathbf{r})$ (with dimensions of radians), which satisfies the equation

$$[\boldsymbol{\nabla}\psi(\mathbf{r})]^2 = k^2(\mathbf{r}) \equiv \left(\frac{\omega n(\mathbf{r})}{c}\right)^2 \ . \tag{27}$$

The unit tangent vector $\mathbf{s}$ to a ray trajectory is defined as

$$\mathbf{s} = \frac{\boldsymbol{\nabla}\psi}{k(\mathbf{r})}, \qquad |\,\mathbf{s}\,| = 1 \ . \tag{28}$$

Subsequently we will use the derivative along the trajectory

$$\frac{d}{dl} \equiv (\mathbf{s} \cdot \boldsymbol{\nabla}) \ . \tag{29}$$

Here and below we assume that some wave front surface is defined by the equation $\psi = \text{const}$. Then Eq. (27) may be solved along the rays:

$$\frac{d}{dl}\psi = k(\mathbf{r}) \ . \tag{30}$$

The trajectories $\mathbf{r}(\mathbf{r_0}, l)$ are defined by the Eq. (1), whereas Eq. (2), or

$$\frac{d\mathbf{s}}{dl} = \mathbf{L} - \mathbf{s}(\mathbf{s} \cdot \mathbf{L}) \ , \tag{31}$$

is the consequence of the definition (28) and eikonal equation (27). The value of div**s** characterizes the rate of change of the "ray tube" cross section S_{tube},

$$\frac{dS_{\text{tube}}}{dl} = -S_{\text{tube}} \text{div}\, \mathbf{s} \ , \tag{32}$$

so that one could expect the energy conservation theorem

$$(|\, A\,|^2 + |\, B\,|^2) S_{\text{tube}} = \text{const} \tag{33}$$

or

$$A, B \sim \exp(-\Gamma), \qquad \frac{d\Gamma}{dl} = \frac{1}{2} \text{div}\, \mathbf{s} \ . \tag{34}$$

Besides, we may choose two circular polarization vectors $\mathbf{a}$ and $\mathbf{a}^*$, which are perpendicular to the local propagation direction [compare with Eqs. (20) and (21)]; they satisfy the conditions

$$\begin{aligned} &(\mathbf{a} \cdot \mathbf{a}) = (\mathbf{a}^* \cdot \mathbf{a}^*) = 0, \\ &(\mathbf{a} \cdot \mathbf{s}) = (\mathbf{a}^* \cdot \mathbf{s}) = 0, \\ &(\mathbf{a} \cdot \mathbf{a}^*) = 1, \end{aligned} \tag{35}$$

$$\mathbf{a} \times \mathbf{s} = i\mathbf{a}, \quad \mathbf{a}^* \times \mathbf{s} = -i\mathbf{a}^*, \quad \mathbf{a}^* \times \mathbf{a} = i\mathbf{s} \ . \tag{36}$$

If we suppose that the vectors $\mathbf{a}$ and $\mathbf{a}^*$ obey the parallel transport equation (8), or

$$\frac{d\mathbf{a}}{dl} = -\mathbf{s}(\mathbf{a} \cdot \mathbf{L}), \qquad \frac{d\mathbf{a}^*}{dl} = -\mathbf{s}(\mathbf{a}^* \cdot \mathbf{L}) \ , \tag{37}$$

then the properties (36) will stay true in the whole three-dimensional space. Besides, the following properties of that system of vectors will be useful in the following:

$$-\mathbf{a}^* \cdot \text{rot}\, \mathbf{a} = \mathbf{a} \cdot \text{rot}\, \mathbf{a}^* = i\frac{1}{2} \text{div}\, \mathbf{s} \ , \tag{38}$$

$$\text{rot}\, \mathbf{s} = \mathbf{s} \times \mathbf{L}, \qquad \mathbf{s} \cdot \text{rot}\, \mathbf{a} = i(\text{div}\mathbf{a} + \mathbf{a} \cdot \mathbf{L}) \ . \tag{39}$$

The particular choice of the field of the vectors $\mathbf{a}, \mathbf{a}^*$ at the "initial" wave front surface is equivalent to some particular choice of a field of "azimuthal" lines on that surface. The change from one azimuthal field to another may be characterized by the local rotation angle $\delta\varphi(u, v)$, where u and v are some coordinates on that surface. In that case $\mathbf{a}_{\text{new}}(u, v) = \exp[i\delta\varphi(u, v)] \cdot \mathbf{a}_{\text{old}}(u, v)$ and $\mathbf{a}^*{}_{\text{new}} = \exp[-i\delta\varphi(u, v)] \cdot \mathbf{a}^*{}_{\text{old}}$. We assume that the azimuth field is chosen in such a way that the derivatives of $\mathbf{a}^*$ and $\mathbf{a}$ are about of inverse size of the inhomogeneity (the latter unfortunately was also denoted by the letter a in this paper). Now we will look for the solution of Eqs. (15) and (16) in the form

$$\begin{aligned} \mathbf{A}(\mathbf{r}) = [&A_0(\mathbf{r})\mathbf{a}(\mathbf{r}) + A_1(\mathbf{r})\mathbf{a}^*(\mathbf{r}) + A_S(\mathbf{r})\mathbf{s}(\mathbf{r})] \\ &\times \exp[i\psi(\mathbf{r}) - \Gamma(\mathbf{r})] \ , \end{aligned} \tag{40}$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = [&B_1(\mathbf{r})\mathbf{a}(\mathbf{r}) + B_0(\mathbf{r})\mathbf{a}^*(\mathbf{r}) + B_S(\mathbf{r})\mathbf{s}(\mathbf{r})] \\ &\times \exp[i\psi(\mathbf{r}) - \Gamma(\mathbf{r})] \ . \end{aligned} \tag{41}$$

Substituting (40) and (41) into (15) and (16) and multiplying the result by $\mathbf{a}^*$, $\mathbf{a}$, and $\mathbf{s}$, we obtain the exact system of six complex equations of first order in the derivatives

$$\begin{aligned} &(\mathbf{s} \cdot \boldsymbol{\nabla})A_0 + i(\mathbf{a}^* \cdot \text{rot}\, \mathbf{a}^*)A_1 + [\mathbf{a}^* \cdot (\mathbf{L} + \boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla})]A_S \\ &\qquad = \frac{1}{2}[(\mathbf{a}^* \cdot \mathbf{G})B_S - (\mathbf{s} \cdot \mathbf{G})B_1] \ , \end{aligned} \tag{42}$$

$$\begin{aligned} &A_1 - \frac{i}{2k}(\mathbf{s} \cdot \boldsymbol{\nabla})A_1 - \frac{1}{2k}(\mathbf{a} \cdot \text{rot}\, \mathbf{a})A_0 \\ &\quad - \frac{i}{2k}[\mathbf{a} \cdot (\mathbf{L} + \boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla})]A_S \\ &\qquad = \frac{i}{4k}[(\mathbf{s} \cdot \mathbf{G})B_0 - (\mathbf{a} \cdot \mathbf{G})B_S] \ , \end{aligned} \tag{43}$$

$$\begin{aligned} &A_S + \frac{i}{k}[\mathbf{a} \cdot (\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) + i(\mathbf{s} \cdot \text{rot}\, \mathbf{a})]A_0 \\ &\quad - \frac{i}{k}[\mathbf{a}^* \cdot (\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) - i(\mathbf{s} \cdot \text{rot}\, \mathbf{a}^*)]A_1 \\ &\qquad = \frac{i}{2k}[(\mathbf{a} \cdot \mathbf{G})B_1 - (\mathbf{a}^* \cdot \mathbf{G})B_0] \ , \end{aligned} \tag{44}$$

$$\begin{aligned} &(\mathbf{s} \cdot \boldsymbol{\nabla})B_0 - i(\mathbf{a} \cdot \text{rot}\, \mathbf{a})B_1 + [\mathbf{a} \cdot (\mathbf{L} + \boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla})]B_S \\ &\qquad = \frac{1}{2}[(\mathbf{a} \cdot \mathbf{G})A_S - (\mathbf{s} \cdot \mathbf{G})A_1] \ , \end{aligned} \tag{45}$$

$$\begin{aligned} &B_1 - \frac{i}{2k}(\mathbf{s} \cdot \boldsymbol{\nabla})B_1 + \frac{1}{2k}(\mathbf{a}^* \cdot \text{rot}\, \mathbf{a}^*)B_0 \\ &\quad - \frac{i}{2k}[\mathbf{a}^* \cdot (\mathbf{L} + \boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla})]B_S \\ &\qquad = \frac{i}{4k}[(\mathbf{s} \cdot \mathbf{G})A_0 - (\mathbf{a}^* \cdot \mathbf{G})A_S] \ , \end{aligned} \tag{46}$$

$$\begin{aligned} &B_S - \frac{i}{k}[\mathbf{a} \cdot (\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) + i(\mathbf{s} \cdot \text{rot}\, \mathbf{a})]B_1 \\ &\quad + \frac{i}{k}[\mathbf{a}^* \cdot (\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) - i(\mathbf{s} \cdot \text{rot}\, \mathbf{a}^*)]B_0 \\ &\qquad = \frac{i}{2k}[(\mathbf{a}^* \cdot \mathbf{G})A_1 - (\mathbf{a} \cdot \mathbf{G})A_0] \ . \end{aligned} \tag{47}$$

Almost everybody (but not us) would say that this system is much more complicated than the original Maxwell system in Cartesian coordinates. However, Eqs. (42)–(47) allow us to produce an asymptotic expansion of the solution.

IV. PARALLEL TRANSPORT

Let us make an assumption (the validity of which is confirmed by subsequent calculations) that the amplitudes A_1,A_S and B_1,B_S are of the first order in the sense of the small parameter (λ/a) in comparison with zero-order amplitudes A_0 and B_0. In that approximation we see that those zero-order amplitudes propagate independently with conserved values along the rays:

$$\frac{dA_0}{dl} \equiv (\mathbf{s}\cdot\boldsymbol{\nabla})A_0(\mathbf{r}) \simeq O(\lambda/a)^2 \ ,$$
$$\frac{dB_0}{dl} \equiv (\mathbf{s}\cdot\boldsymbol{\nabla})B_0(\mathbf{r}) \simeq O(\lambda/a)^2 \ , \qquad (48)$$

as it should be in standard scalar geometric optics.

V. EFFECTIVE BIREFRINGENCE

Equations (43), (44), (46), and (47) allow us to find the values of A_1, A_S, B_1, and B_S up to an accuracy of first order in (λ/a):

$$A_1 = \frac{1}{2k}(\mathbf{a}\cdot\mathrm{rot}\,\mathbf{a})A_0 + \frac{i}{4k}(\mathbf{G}\cdot\mathbf{s})B_0 \ ,$$
$$A_S = -\frac{i}{k}[\mathbf{a}\cdot(\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) + i(\mathbf{s}\cdot\mathrm{rot}\,\mathbf{a})]A_0 - \frac{i}{2k}(\mathbf{G}\cdot\mathbf{a}^*)B_0 \ , \qquad (49)$$

$$B_1 = -\frac{1}{2k}(\mathbf{a}^*\cdot\mathrm{rot}\,\mathbf{a}^*)B_0 + \frac{i}{4k}(\mathbf{G}\cdot\mathbf{s})A_0 \ ,$$
$$B_S = -\frac{i}{k}[\mathbf{a}^*\cdot(\boldsymbol{\nabla}\Gamma - \boldsymbol{\nabla}) - i(\mathbf{s}\cdot\mathrm{rot}\,\mathbf{a}^*)]B_0 - \frac{i}{2k}(\mathbf{G}\cdot\mathbf{a})A_0 \ . \qquad (50)$$

Back substitution of those expressions into Eqs. (42) and (45) allows us to get the coupled equations for "large" amplitudes A_0 and B_0 only, but now with the accuracy $\sim (\lambda/a)^2$ included.

We are not going to study all the terms of that order in this paper. Among other effects, they include transverse diffraction in parabolic approximation and the optical magnus effect. Concentrating on the effects of linear birefringence, we will consider here only the terms $\sim G$ which lead to the coupling between A_0 and B_0. In that approach, after rather lengthy calculations, one obtains the quite simple looking result

$$\frac{dA_0}{dl} = -\frac{i}{2k}B_0\cdot[\mathbf{a}^*(\mathbf{a}^*\cdot\boldsymbol{\nabla})\mathbf{G} - 2(\mathbf{G}\cdot\mathbf{a}^*)(\mathbf{L}\cdot\mathbf{a}^*)] \ , \qquad (51)$$

$$\frac{dB_0}{dl} = -\frac{i}{2k}A_0\cdot[\mathbf{a}(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{G} - 2(\mathbf{G}\cdot\mathbf{a})(\mathbf{L}\cdot\mathbf{a})] \ . \qquad (52)$$

To compare this with the "usual" birefringence, we suppose for a moment that the tensors of dielectric and magnetic susceptibilities are not as in Eq. (3), but instead they are equal to

$$\hat\varepsilon_{ik} = \varepsilon\delta_{ik} + \delta\hat\varepsilon_{ik}, \quad \hat\mu_{ik} = \mu\delta_{ik} + \delta\hat\mu_{ik} \ . \qquad (53)$$

In that case the basic equations take the form (in first order in $\delta\hat\varepsilon$ and $\delta\hat\mu$)

$$\mathrm{rot}\mathbf{A} = k\left(1+\frac{\delta\hat\varepsilon}{2\varepsilon}+\frac{\delta\hat\mu}{2\mu}\right)\mathbf{A} - \frac{1}{2}\mathbf{G}\times\mathbf{B} + k\left(\frac{\delta\hat\varepsilon}{2\varepsilon}-\frac{\delta\hat\mu}{2\mu}\right)\mathbf{B} \ , \qquad (54)$$

$$\mathrm{rot}\mathbf{B} = -k\left(1+\frac{\delta\hat\varepsilon}{2\varepsilon}+\frac{\delta\hat\mu}{2\mu}\right)\mathbf{A} - \frac{1}{2}\mathbf{G}\times\mathbf{A} - k\left(\frac{\delta\hat\varepsilon}{2\varepsilon}-\frac{\delta\hat\mu}{2\mu}\right)\mathbf{A} \ . \qquad (55)$$

Then neglecting the G term we obtain

$$\frac{dA_0}{dl} - i\delta k_0 A_0 = ikB_0\cdot\left[\mathbf{a}^*\cdot\left(\frac{\delta\hat\varepsilon}{2\varepsilon}-\frac{\delta\hat\mu}{2\mu}\right)\cdot\mathbf{a}^*\right] \ , \qquad (56)$$

$$\frac{dB_0}{dl} - i\delta k_0 B_0 = ikA_0\cdot\left[\mathbf{a}\cdot\left(\frac{\delta\hat\varepsilon}{2\varepsilon}-\frac{\delta\hat\mu}{2\mu}\right)\cdot\mathbf{a}\right] \ . \qquad (57)$$

Here δk_0 is the isotropic addition to the wave vector

$$\delta k_0 = k\mathbf{a}^*\cdot\left(\frac{\delta\hat\varepsilon}{2\varepsilon}+\frac{\delta\hat\mu}{2\mu}\right)\cdot\mathbf{a} \ . \qquad (58)$$

Since we assume that $\delta\hat\varepsilon$ and $\delta\hat\mu$ are symmetric tensors, we may rewrite (58) as

$$\delta k_0 = \frac{k}{4}\mathrm{Tr}\left[\left(\frac{\delta\hat\varepsilon}{2\varepsilon}+\frac{\delta\hat\mu}{2\mu}\right)(\hat 1 - \mathbf{s}\otimes\mathbf{s})\right] \ , \qquad (59)$$

where $(\mathbf{s}\otimes\mathbf{s})_{ik} = s_i s_k$ and Tr stands for the trace of the tensor. Comparing Eq. (50) with (54) we see that the birefringence due to smooth gradients of medium properties is equivalent to

$$\left(\frac{\delta\hat\varepsilon}{\varepsilon}-\frac{\delta\hat\mu}{\mu}\right)_{ik} = \frac{1}{k^2}\left(\frac{\partial^2\ln\rho}{\partial x_i\partial x_k} - \frac{\partial\ln\rho}{\partial x_i}\frac{\partial\ln n}{\partial x_k} - \frac{\partial\ln\rho}{\partial x_k}\frac{\partial\ln n}{\partial x_i}\right) \ . \qquad (60)$$

This is the main result of the present paper.

VI. DISCUSSION

First of all, one sees from (60) that the effective birefringence vanishes if ρ = const in the medium, i.e., if $\mathbf{G} = \boldsymbol{\nabla}\ln\rho \equiv \mathbf{0}$. However, for the case when n =const the birefringence may be nonzero due to the term $\sim (\partial^2\ln\rho/\partial x_i\partial x_k)$. Moreover, if the changes of ε, μ, n, and ρ are rather small, $\delta\varepsilon \ll \varepsilon$, $\delta\mu \ll \mu$, then the terms $\sim (\partial\ln\rho/\partial x_i)(\partial\ln n/\partial x_k)$ are smaller than the second derivative term in Eq. (60) by a factor $(\delta n/n)$.

It is interesting to compare Eq. (60) with the result for effective birefringence obtained earlier in [16] for the one-dimensional case (layered medium), $n = n(z)$, $\rho = \rho(z)$. The calculation of the phase difference between the $Ee^{ik_x x}\mathbf{e}_y$ polarized wave and the $He^{ik_x x}\mathbf{e}_y$ polarized one using Eq. (60) allows us to obtain

$$(\varphi_E - \varphi_H) = \int_{-\infty}^{+\infty} dl \frac{c\beta^2}{2\omega n^3\gamma^2} \frac{d\ln n}{dz}\frac{d\ln\rho}{dz}$$
$$\equiv \int_{-\infty}^{+\infty} dl \frac{c\beta^2}{2\omega n}\left(\frac{d^2\ln\rho}{dz^2} - 2\frac{d\ln n}{dz}\frac{d\ln\rho}{dz}\right)$$
$$\equiv \int_{-\infty}^{+\infty} dl \frac{c}{2\omega}\left(\frac{d^2\ln\rho}{dz^2} - \frac{d\ln n}{dz}\frac{d\ln\rho}{dz}\right) . \quad (61)$$

Here $k_x = \omega\beta/c =$const,

$$\beta = \sin\alpha_{\rm air}, \quad \gamma(z) = \sqrt{1-\beta^2/n^2(z)} = \cos\alpha(z) \ , \quad (62)$$

where $\alpha(z)$ is the current angle between the propagation direction and z axis, so that $dz/dl = \gamma(z)$. We see that the expression following from the present paper's result (60), i.e., the second line of the Eq. (61), is identical to the one-dimensional result from [16], i.e., to the first and third lines.

For the one-dimensional case birefringence is accumulated along the ray if both the refractive index and impedance are inhomogeneous. Beside that, it means that $\delta\varphi \sim (\delta n/n)^2(\lambda/a)$ for the one-dimensional case, as opposed to the general three-dimensional situation. It is worth mentioning that $\mu \equiv 1$ in optics of visible light, and therefore $\ln\rho(\mathbf{r}) \equiv -\ln n(\mathbf{r})$.

The estimation of the size $a \sim 10(\lambda/2\pi n) = 1\ \mu$m at $\lambda_0 = 0.6\ \mu$m and $n = 1.5$ and for refractive index step $\delta n = 0.15$ gives $\delta\varphi \sim (\delta n/n)^2(\lambda/a) \sim 10^{-3}$ rad. We hope that such a phase difference can be easily measured by the methods of modern ellipsometry [18–20]. A smooth gradient of the refractive index may be obtained, e.g., via diffusion through a boundary between two mutually solvable liquids with different refractive indices.

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